A new algorithm for the effective Deuring correspondence: making SQISign faster.

Antonin Leroux, joint work with Luca De Feo, Patrick Longa, Benjamin Wesolowski

Isogeny Club, October 25, 2022

DGA, France

The Deuring correspondence

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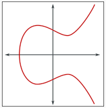

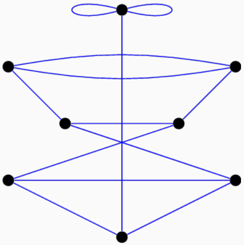

Credits to Luca De Feo and Cmglee

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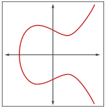

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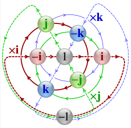

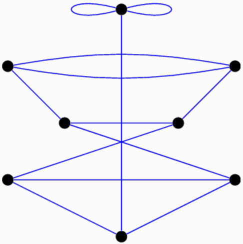

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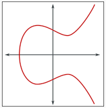

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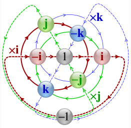

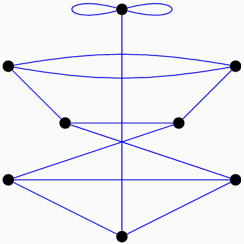

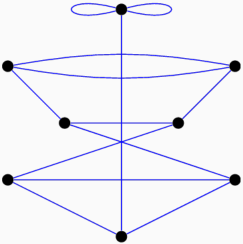

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The rest of this talk

The plan:

- Introduction to the Deuring correspondence
- Algorithmic aspects: theory.
- Algorithm aspects: practice, the ideal to isogeny translation.
- Application to SQISign.

Mathematical Background

Quaternion algebra definitions

The quaternion algebra $\mathcal{B}(a, b)$ over $\mathbb{Q}$ with $a, b \in \mathbb{Z}$ is

$$\mathcal{B}(a, b) = \mathbb{Q} + i\mathbb{Q} + j\mathbb{Q} + k\mathbb{Q}$$

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Orders are rings: so we have ideals. In a non-commutative algebra, ideals have distinct left and right orders.

There is $n : \mathcal{B}(a, b) \to \mathbb{Q}$, and the norm is integral over orders, so we can define **ideal norm** as $\{\gcd(n(\alpha)), \alpha \in I\}$.

Elliptic curve and isogeny notations

Elliptic Curve over $\mathbb{F}_{p^k}$:

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An **endomorphism** is an isogeny $\varphi : E \to E$. $End(E)$ is a ring.

Supersingular curves $\Leftrightarrow$ $\mathrm{End}(E)$ is a max. *order* in a quaternion algebra.

The Deuring Correspondence

p : prime characteristic, $\mathcal{B}(-q,-p)$ where $q > 0$ depends only on p.

Supersingular elliptic curves over $\mathbb{F}_{p^2}$ E (up to Galois conjugacy)	Maximal Orders in $\mathcal{B}(-q,-p)$ $\mathcal{O} \cong \mathrm{End}(E)$
Isogeny with $\varphi : E \to E_1$	Ideal I_φ left $\mathcal{O}$-ideal
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$$\mathrm{End}(E_0) = \langle 1, \iota, \frac{\iota + \pi}{2}, \frac{1 + \iota\pi}{2} \rangle \cong \langle 1, i, \frac{i + j}{2}, \frac{1 + k}{2} \rangle$$

$\pi : (x, y) \mapsto (x^p, y^p)$ is the Frobenius morphism with $\pi \circ \pi = [-p]$.

$\iota : (x, y) \mapsto (-x, \sqrt{-1}y)$ is a twisting automorphism with $\iota \circ \iota = [-1]$.

Kernel ideals

Let $\varphi : E \to E'$ be an isogeny of degree D. The kernel ideal I_φ of I is defined as

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Alternatively, we have

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Conversely, the kernel of an $\mathcal{O}$-ideal I (for $\mathcal{O} \cong \mathrm{End}(E)$)

$$E[I] = \{P, \alpha(P) = 0 \text{ for all } \alpha \in I\} = \bigcap_{\alpha \in I} \ker \alpha.$$

We define $\varphi_I : E \to E/E[I]$.

A new hard problem?

Supersingular ℓ-Isogeny Problem: Given a prime p and two supersingular curves E_1 and E_2 over $\mathbb{F}_{p^2}$, compute an ℓ^e-isogeny $\varphi : E_1 \to E_2$ for $e \in \mathbb{N}^\star$.

Quaternion ℓ-Isogeny Path Problem: Given a prime number p, two maximal orders $\mathcal{O}_1, \mathcal{O}_2$ of $\mathcal{B}(-q, -p)$, find an ideal J of norm ℓ^e for $e \in \mathbb{N}^\star$ with $\mathcal{O}_L(J) \cong \mathcal{O}_1$, $\mathcal{O}_R(J) \cong \mathcal{O}_2$.

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[EHLMP18; W22]: use KLPT to prove *polynomial-time* reduction from supersingular ℓ-isogeny problem to:

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Ideal to isogeny: theory vs practice.

Ideal to isogeny translation

Input: A supersingular *curve* E, a *maximal order* $\mathcal{O}$ with $\mathcal{O} \cong \mathrm{End}(E)$, and an $\mathcal{O}$*-ideal* I of norm D (both given as 16 coefficients over $\mathcal{B}(-q,-p)$).

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Motivation: make the computation efficient in practice for a big smooth degree D (application to SQISign).

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An algorithm from Galbraith, Petit and Silva [GPS17]:

1. Evaluate the elements of $I \hookrightarrow \mathrm{End}(E)$ on the D-torsion.
2. Find the common kernel $E[I]$ (DLP computations)
3. Compute φ_I from $\ker \varphi_I = G$.

Complexity: polynomial in some nice cases...

Generalizing the approach

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For 1: Factor φ_I and apply the algorithm on the factor isogenies of small degrees. This means several intermediate curves: we really need to find a solution to 2.

For 2...

Evaluating the elements of an arbitrary endo. ring: a first approach

$\mathcal{O} \cong \mathsf{End}(E)$, a point $P \quad \rightarrow \quad \alpha(P)$ for some $\alpha \in \mathsf{End}(E)$.

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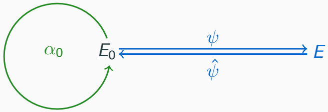
$$\alpha = \frac{1}{[\deg \psi]} \psi \circ \alpha_0 \circ \hat{\psi}$$



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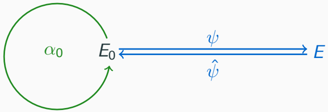

1. Compute $\psi : E_0 \rightarrow E$ with KLPT and the algorithm from [GPS17] (need $T = \deg \psi$ coprime to D!).
2. Express α from ψ and some $\alpha_0 \in \mathrm{End}(E_0)$ (lollipop endomorphism).
3. Evaluate ψ, α_0 to derive $\alpha(P)$.

Evaluating the elements of an arbitrary endo. ring: improvement.

$\mathcal{O} \cong \mathrm{End}(E)$, a point $P \quad \rightarrow \quad \alpha(P)$ for some $\alpha \in \mathrm{End}(E)$.

[F**L**LW22]: we can restrict to α of smooth norm T coprime with D.
Idea: if α is in the Eichler order $\mathrm{End}(E_0) \cap \mathrm{End}(E)$, we will first find the version of $\alpha \in \mathrm{End}(E_0)$ and then use an isogeny $\varphi : E_0 \rightarrow E$ to compute the version in $\mathrm{End}(E)$. If $n(\alpha)$ is coprime with D, φ can be the isogeny we are translating!

1. Compute $\alpha \in \mathcal{B}(-p, -q)$ of smooth norm in $\mathrm{End}(E_0) \cap \mathrm{End}(E)$.
2. Compute α as an isogeny in $\mathrm{End}(E_0)$ from its kernel.
3. Compute α as an isogeny in $\mathrm{End}(E)$ from its kernel with $\varphi : E_0 \rightarrow E$.
4. Evaluate $\alpha(P)$.

KLPT and generalizations

Goal of KLPT: find an ideal of smooth norm connecting two maximal orders $\mathcal{O}_1, \mathcal{O}_2$. Takes another connecting ideal I in input.

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The second algorithm is better because smaller torsion requirement.

A specific choice of parameters

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We sieve through families of primes where a portion of the torsion requirement is forced.

A smaller T helps a lot finding a good smoothness bound on T.

Example: p_{6983} vs p_{3923}

For algorithm 1 we have p_{6983}

$$\begin{aligned} p+1 &= 2^{33} \cdot 5^{21} \cdot 7^2 \cdot 11 \cdot 31 \cdot 83 \cdot 107 \cdot 137 \cdot 751 \cdot 827 \cdot 3691 \cdot 4019 \cdot 6983 \\ &\quad \cdot 517434778561 \cdot 26602537156291\,, \\ p-1 &= 2 \cdot 3^{53} \cdot 43 \cdot 103^2 \cdot 109 \cdot 199 \cdot 227 \cdot 419 \cdot 491 \cdot 569 \cdot 631 \cdot 677 \cdot 857 \cdot 859 \\ &\quad \cdot 883 \cdot 1019 \cdot 1171 \cdot 1879 \cdot 2713 \cdot 4283 \end{aligned}$$

For algorithm 2 we have p_{3923}

$$\begin{aligned} p+1 &= 2^{65} \cdot 5^2 \cdot 7 \cdot 11 \cdot 19 \cdot 29^2 \cdot 37^2 \cdot 47 \cdot 197 \cdot 263 \cdot 281 \cdot 461 \cdot 521 \\ &\quad \cdot 3923 \cdot 62731 \cdot 96362257 \cdot 3924006112952623\,, \\ p-1 &= 2 \cdot 3^{65} \cdot 13 \cdot 17 \cdot 43 \cdot 79 \cdot 157 \cdot 239 \cdot 271 \cdot 283 \cdot 307 \cdot 563 \cdot 599 \\ &\quad \cdot 607 \cdot 619 \cdot 743 \cdot 827 \cdot 941 \cdot 2357 \cdot 10069\,. \end{aligned}$$

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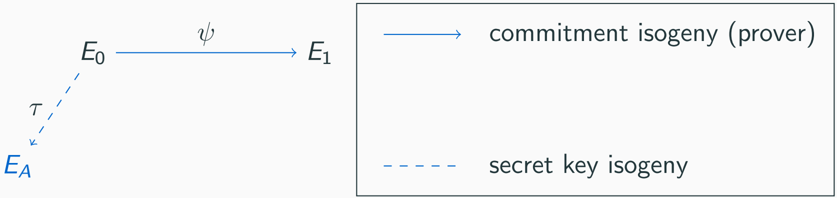

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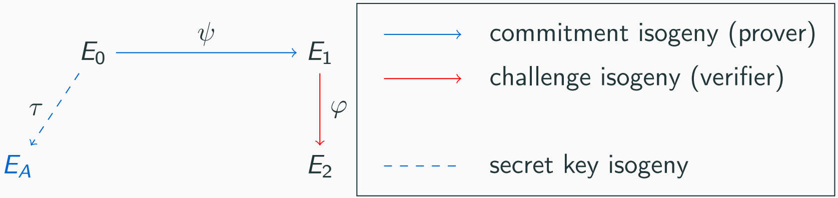

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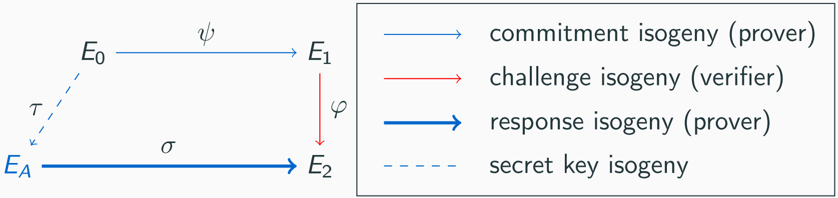

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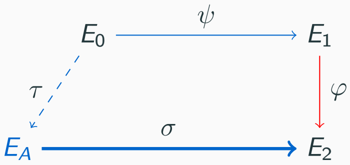

⟶ (thin blue)	commitment isogeny (prover)
⟶ (thin red)	challenge isogeny (verifier)
⟶ (thick blue)	response isogeny (prover)
- - - - -	secret key isogeny

Response computation:

1. Compute $\mathrm{End}(E_2)$ from ψ, φ.
2. Apply KLPT to compute I_σ connecting $\mathrm{End}(E_A)$ and $\mathrm{End}(E_2)$. For security, need generic version of the algorithm!
3. Translate I_σ into σ.

SQISign: Short Quaternion Isogeny Signature

Most compact PQ signature scheme with PK + Signature combined.

Name	Public Key (bytes)	Signature (bytes)	Security
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Non-standard security assumption but safe from recent attacks!

Future work and open problems

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- Find constructive applications of the new attacks (on-going work).