Torsion point images in SIDH: from savior to killer

Tako Boris Fouotsa, LASEC-EPFL

Isogeny Club, 8th November 2022

Introduction 1/3

Isogeny-Based cryptography: very compact keys and mathematically elegant.
But: young field, relatively slow.

First key exchange: CRS[1], uses ordinary isogenies.

Two main issues with using ordinary isogenies:

1. Small amount of smooth rational kernel
2. Arise from class group action : quantum sub-exponential time (CJS 2014)

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Good news:

1. Large amount of rational torsion available (special primes)
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Bad news: supersingular isogenies do not commute !

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dQKL+ 2021: improvement on Petit TPA, but SIDH still safe.

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CD-MM-R 2022, final shot: SIDH/SIKE is broken in seconds...

All these attacks exploit torsion point information !!

Non exhaustive list: BdQL+ 2019, ...

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Outline

The role of torsion points in SIDH

GPST adaptive attack on SIDH

A framework for torsion point attacks

A new adaptive attack on SIDH

Summary

The role of torsion points in SIDH

Recall

Elliptic curve $E/\mathbb{F}_q$: abelian group structure, n-torsion group for n $(p \nmid n)$

$$E[n] = \langle P, Q \rangle \simeq \mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$$

Supersingular curves:

- $\mathrm{End}(E) \simeq \mathcal{O}_{\max} \subset \mathcal{B}_{p,\infty}$
- defined over $\mathbb{F}_{p^2}$ and $E(\mathbb{F}_{p^2}) \simeq \mathbb{Z}/(p \pm 1)\mathbb{Z} \oplus \mathbb{Z}/(p \pm 1)\mathbb{Z}$
- Smooth order when $p \pm 1$ is smooth

Supersingular cyclic d-isogenies:

- do not commute
- can be defined by a scalar α where $\ker \phi = \langle P + [\alpha]Q \rangle$ and $E[d] = \langle P, Q \rangle$.

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E_0 \arrow[r, "\phi_A"] \arrow[dd, "\phi_B"'] & E_A \arrow[d, "\phi'_B"] \\
 & E_{AB} \arrow[d, phantom, "\neq"] \\
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How would you define ϕ'_A and ϕ'_B? Will the resulting diagram commute?

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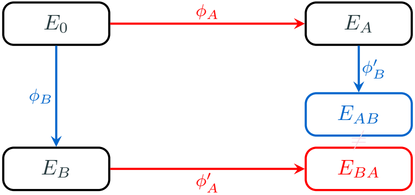

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Validation method: $e_{N_A}(\phi_B(P_A), \phi_B(Q_A)) = e_{N_A}(P_A, Q_A)^{N_B}$

SSI-T Problem: Given E_0, P_B, Q_B, E_A, $\phi_A(P_B)$, $\phi_A(Q_B)$, compute ϕ_A.

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GPST adaptive attack on SIDH

An overview

key exchange oracle:

$$O(E, R, S, E') = \begin{cases} 1 & \text{if } E/\langle R + [\alpha]S \rangle = E' \\ 0 & \text{if } E/\langle R + [\alpha]S \rangle \neq E' \end{cases}$$

Idea of the attack: recursively

- Add some well calibrated noise in the TP images
- Use the key exchange oracle determine if the noise was erased during the key exchange or not.
- Deduce a bit of α.

Adding noise := scaling the TP images by a 2×2 matrix M_i

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The GPST: recovering α with $N_A = 2^a$

Parity of α : use $M_1 = \begin{bmatrix} 1 & 0 \\ 2^{a-1} & 1 \end{bmatrix}$,

$R_1 = \phi_B(P_A)$ and $S_1 = \phi_B(Q_A) + [2^{a-1}]\phi_B(P_A)]$.

$$\begin{aligned} O(E_B, R_1, S_1, E_{AB}) = 1 \quad &\Leftrightarrow \quad E/\langle R_1 + [\alpha]S_1 \rangle = E_{AB} \\ &\Leftrightarrow^* \quad \langle R_1 + [\alpha]S_1 \rangle = \langle \phi_B(P_A) + [\alpha]\phi_B(Q_A) \rangle \\ &\Leftrightarrow \quad \alpha \text{ is even} \end{aligned}$$

Continuing the attack : write $\alpha = K_i + 2^i\alpha_i + 2^{i+1}\alpha'$.

Use $M_i = \theta \begin{bmatrix} 1 & -2^{a-i-1}K_i \\ 0 & 1 + 2^{a-i-1} \end{bmatrix}$, where $\theta = \sqrt{(1 + 2^{a-i-1})^{-1}}$.

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A framework for torsion point attacks

More facts about isogenies

- For any seperable d-isogeny $\varphi : E \to E'$, there exist a unique* d-isogeny $\hat{\varphi} : E' \to E$ called the dual the dual of φ such that $\hat{\varphi} \circ \varphi = [d]_E$ and $\varphi \circ \hat{\varphi} = [d]_{E'}$.
- We have

$$\ker \hat{\varphi} = \varphi(E[d]) \quad \text{and} \quad \ker \varphi = \hat{\varphi}(E'[d]).$$

Take away:

- The knowledge of φ is equivalent to the knowledge of $\hat{\varphi}$.
- You can recover the kernel of a d-isogeny φ by evaluating φ on the d-torsion group.

The framework

SSI-T Problem: Given E_0, $E[N_B] = \langle P, Q \rangle$, E, $\phi(P)$, $\phi(Q)$, compute ϕ.

Degree transformation: define a map Γ that can be used to transform ϕ to $\tau = \Gamma(\phi, input)$ such that:

1. Knowing $\tau = \Gamma(\phi, input)$, one can recover ϕ
2. τ can be evaluated on the N_B-torsion
3. τ can be recovered from its action on the N_B-torsion

The attack: Given a suitable description of Γ,

- Use 2. and 3. to recover τ
- Use 1. to derive ϕ from τ

Petit2017 and dQKL+2021

Assumes that $\mathrm{End}(E_0)$ is known. $input = [\theta \in End(E_0),\, d \in \mathbb{Z}]$.

$$\tau = \Gamma(\phi, \theta, d) := [d] + \phi \circ \theta \circ \hat{\phi}$$

s.t. $\deg \tau = N_B^2 e$ with e small.

$$\begin{tikzcd} E_0 \arrow[loop left, "\theta"] \arrow[r, shift left, "\phi"] & E \arrow[l, shift left, "\hat{\phi}"] \arrow[loop right, "\tau"] \end{tikzcd}$$

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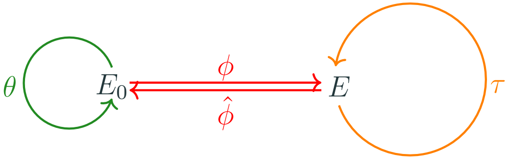

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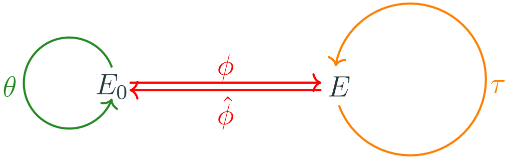

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$\ker \hat{\phi} =^* \ker(\tau - [d]) \cap E[N_A]$

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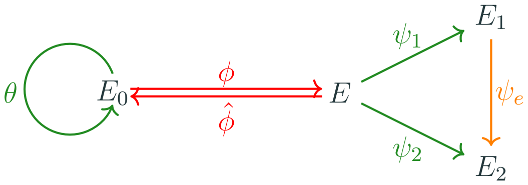

- ψ_1 and ψ_2 can be computed from $\phi(P), \phi(Q)$.
- ψ_e is recovered by brute force.

Easy to find good $[\theta \in End(E_0),\ d \in \mathbb{Z}]$ when $N_B > pN_A$.

SIDH : $N_A \approx N_B \approx \sqrt{p}$. Still Secure !

dQKL+2021: τ from its action on the N_B-torsion

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CD-MM-R 2022

Assume $\phi : E_0 \longrightarrow E_B$ has degree N_B and the TP have order N_A. Set $a = N_A - N_B = a_1^2 + a_2^2 + a_3^2 + a_4^2$.

$$\tau = \Gamma(\phi, a) := \begin{bmatrix} \alpha_0 & \hat{\phi} Id_4 \\ -\phi Id_4 & \hat{\alpha}_B \end{bmatrix} \in \mathrm{End}(E_0^4 \times E_B^4)$$

where

- $\phi Id_4 : E_0^4 \longrightarrow E_B^4$
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- $\alpha_0 \in \mathrm{End}(E_0^4)$ and $\alpha_B \in \mathrm{End}(E_B^4)$ having the same matrix representation

$$M = \begin{bmatrix} a_1 & -a_2 & -a_3 & -a_4 \\ a_2 & a_1 & a_4 & -a_3 \\ a_3 & -a_4 & a_1 & a_2 \\ a_4 & a_3 & -a_2 & a_1 \end{bmatrix}$$

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Fact: τ has degree $N_B + a = N_A$

1. Knowing $\tau = \Gamma(\phi, input)$, one can recover ϕ
2. τ can be evaluated on the N_A-torsion
3. τ can be recovered from its action on the N_A-torsion

Runs in polynomial time !! Breaks SIDH/SIKE/SETA/...

Countermeasures? ongoing...

More details: eprint 2022/1038 or Lorenz's blog post

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A new adaptive attack on SIDH

key exchange oracle:

$$O(E,R,S,E') = \begin{cases} 1 & \text{if } E/\langle R+[\alpha]S\rangle = E' \\ 0 & \text{if } E/\langle R+[\alpha]S\rangle \neq E' \end{cases}$$

Idea of the attack

1 Actively (using the key exchange oracle) recover the action of ϕ_A on large pairwise disjoint cyclic groups $G_1, G_2, G_3 \subset E_0[NN_B]$ of order NN_B where $p < N$.

2 Use torsion point attacks to recover ϕ_A.

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Step 1: Recovering the action of ϕ_A on a larger group

Set $N = \prod_{i=1}^{e} \ell_i^2$, $\quad \ell_i$ coprime to $N_A N_B$.

Let $\ell \mid N$ and let G be a cyclic group of order $\ell^2 N_B$.

Query: $O(E_G, R, S, E_H)$, $R = [\ell^{-1}]\phi_G(P_A)$, $S = [\ell^{-1}]\phi_G(Q_A)$

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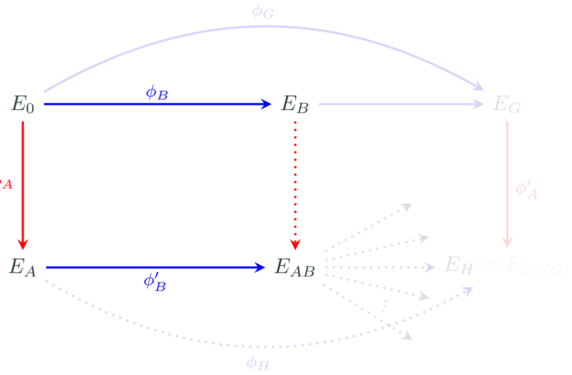

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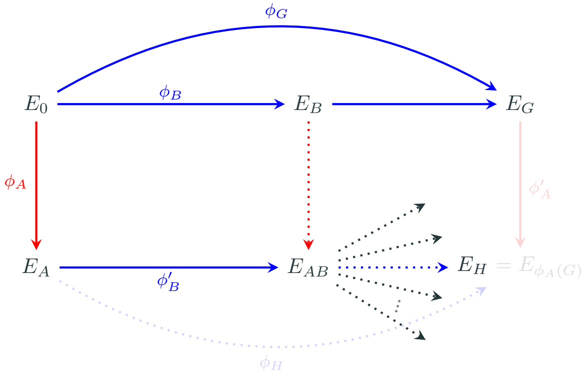

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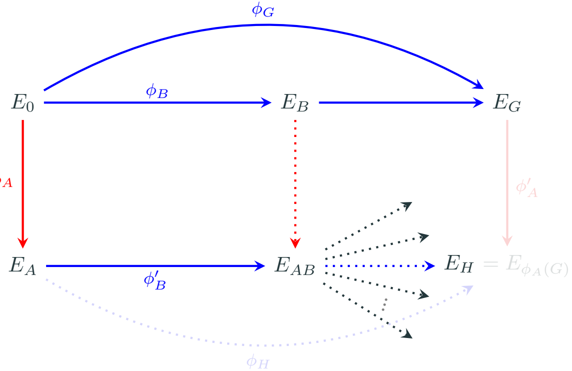

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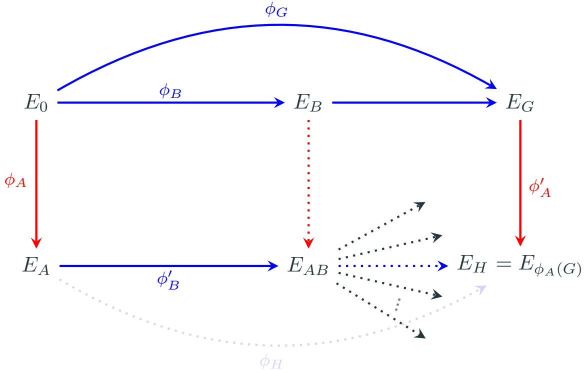

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Countermeasures

- Start from a supersingular curve E_0 with unknown endomorphism ring, this would counter the torsion point attacks that are used as subroutine in the attack.
- Use FO-transform as in SIKE: when running the re-encryption step in the FO, Alice will notice that the public key used was malicious.

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Torsion points have caused the dead of SIDH/SIKE. Any hope for countermeasures? May be:

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- Masked torsion points SIDH? (F. 2022)

Current analysis shows that the primes used should have at least ≈ 6000 bits !

Moreover, they are still vulnerable to adaptive attacks. So would still require FO to have IND-CCA security

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Happy to discuss your comments and questions !!!